

The naturality of natural deduction

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Abstract

Developing a suggestion by Russell, Prawitz showed how the usual natural deduction inference rules for disjunction, conjunction and absurdity can be derived using those for implication and the second order quantifier in propositional intuitionistic second order logic NI^2 . It is however well known that the translation does not preserve the relations of identity among derivations induced by the permutative conversions and immediate expansions for the definable connectives, at least when the equational theory of NI^2 is assumed to consist only of β and η equations. On the basis of the categorical interpretation of NI^2 , we introduce a new class of equations expressing what in categorical terms is a naturality condition satisfied by the transformations interpreting NI^2 -derivations. We show that the Russell-Prawitz translation does preserve identity of proof with respect to the enriched system by highlighting the fact that naturality corresponds to a generalized permutation principle. We show that these result generalize some facts which have gone so far unnoticed, namely that the Russell-Prawitz translation maps particular classes of instances of the equations governing disjunction (and the other definable connectives) onto equations which are already included in the $\beta\eta$ -equational theory of NI^2 . Finally, we compare our approach with the one proposed by Ferreira and Ferreira and show that the naturality condition suggests a generalization of their methods to a wider class of formulas.

Keywords Dinaturality condition, permutative conversions, η -conversion, Russell-Prawitz translation, second order logic

1 Introduction

Since Russell [16] it is known that in propositional second order logic it is possible to define disjunction (as well as conjunction and absurdity) using implication and the second order quantifier. In his doctoral dissertation [13], Prawitz showed how the usual natural deduction inference rules for disjunction (as well as for conjunction and absurdity) can be derived using those for implication and the second order quantifier in

propositional intuitionistic second order logic. Following [5, 6] (but see also [1]) we refer to the derivability-preserving translation of propositional intuitionistic second order logic $\text{NI}^{2\vee}$ into its disjunction- (and conjunction- and absurdity-) free fragment NI^2 as the “Russell-Prawitz translation”.

The conversions used to establish the normalization results for natural deduction systems can be viewed as inducing an equational theory on derivations [14]. One may therefore expect that the Russell-Prawitz translation preserves the relations of equivalence among derivations. However, this is not the case in general, at least when one considers the usual equational theory for NI^2 , i.e. the one consisting of the so called β - and η -equations. In particular, although β -equivalent derivations in the full language are mapped onto β -equivalent derivations in the implicational fragment, the same does not happen for η -equivalent derivations, nor for derivations which are equivalent modulo the equations corresponding to the permutative (or commutative) conversions for disjunction (we will refer to these as γ -equations).

In categorical interpretations of propositional intuitionistic second order logic (see [10]), formulas are interpreted as particular functors, and derivations are viewed as *natural transformations* between these functors. The naturality of the transformations can be expressed as a particular class of equations. Generalizing these equations results in an extension of the $\beta\eta$ -equational theory for NI^2 , and the Russell-Prawitz translation maps γ -equations onto a particular sub-class of these new equations.

With the goal of making these results accessible to a wider community, we will give a purely proof-theoretical presentations of them. In particular, we will highlight the fact that the naturality condition, in terms of natural deduction, corresponds to a general permutation principle.

We will show that our results are actually a generalization of some elementary facts which have gone so far unnoticed, namely that particular classes of instances of the γ - and η -equations in $\text{NI}^{2\vee}$ are mapped by the Russell-Prawitz translation onto equations which are included in the $\beta\eta$ -equational theory of NI^2 .

In a recent series of papers Ferreira and Ferreira [5, 6] advanced another approach to solve the problem of preserving η - and γ -equations in $\text{NI}^{2\vee}$ (and its extensions with the other definable connectives). The main ingredient of their approach is a result (that they call *instantiation overflow*) which holds for the fragment NI_{at}^2 of NI^2 enjoying the sub-formula property obtained by restricting the elimination rule for the second order quantifier $\forall E$ to atomic substitution: in NI_{at}^2 an unrestricted applications of $\forall E$ is derivable provided that the premise of the rule application has the form of the Russell-Prawitz translation of some propositional formula. As we show in the last section, the naturality condition endows Ferreira and Ferreira’s result of a categorical content and suggests moreover to generalize it to a wider class of formulas.

2 Preliminaries

Given a countable set of propositional variables \mathcal{V} , the formulas $\Phi^{2\vee}$ of the language $\mathcal{L}^{2\vee}$ are defined by the following grammar:

$$\Phi^{2\vee} ::= \mathcal{V} \mid (\forall \mathcal{V} \Phi^{2\vee}) \mid (\Phi^{2\vee} \supset \Phi^{2\vee}) \mid (\Phi^{2\vee} \vee \Phi^{2\vee}).$$

As usual, we omit outermost parentheses and we take iterated implications to associate to the right. We call \mathcal{L}^\vee the restriction of $\mathcal{L}^{2\vee}$ to the $\{\supset, \vee\}$ language fragment and \mathcal{L}^2 the restriction of $\mathcal{L}^{2\vee}$ to the $\{\supset, \forall\}$ language fragment. By $A[B/X]$ we indicate the result of substituting the formula B for the variable X in A .

We define the natural deduction system $\text{NI}^{2\vee}$ as follows:¹

Definition 2.1 ($\text{NI}^{2\vee}$ -derivation).

- For all formulas A of $\mathcal{L}^{2\vee}$ and natural number n , $\overset{n}{A}$ is a $\text{NI}^{2\vee}$ -derivation of A from undischarged assumption A ;
- if \mathcal{D}' , \mathcal{D}'' , \mathcal{D}_1 and \mathcal{D}_2 are $\text{NI}^{2\vee}$ -derivations then the following:

$$\begin{array}{c} \frac{\mathcal{D}'}{\frac{A}{\forall X A}} \forall I \quad \text{(provided } X \text{ doesn't occur free in the assumptions of } \mathcal{D}_1\text{)} \\[10pt] \frac{\overset{n}{[A]} \quad \mathcal{D}_1}{\frac{B}{A \supset B}} \supset I(n) \\[10pt] \frac{\mathcal{D}'}{\frac{A}{A \vee B}} \vee I_1 \quad \frac{\mathcal{D}'}{\frac{B}{A \vee B}} \vee I_2 \\[10pt] \frac{\mathcal{D}'}{\frac{\forall X A}{A[B/X]}} \forall E \quad \frac{\mathcal{D}' \quad \mathcal{D}''}{\frac{A \supset B \quad A}{B}} \supset E \\[10pt] \frac{\mathcal{D}' \quad \overset{n}{[A]} \quad \overset{m}{[B]} \quad \mathcal{D}_1 \quad \mathcal{D}_2}{\frac{A \vee B \quad C \quad C}{C}} \vee E(n, m) \end{array}$$

are $\text{NI}^{2\vee}$ -derivations as well, where the assumptions $\overset{n}{A}$ (resp. $\overset{m}{B}$) that are undischarged in \mathcal{D}_1 (resp. in \mathcal{D}_2) become discharged in the derivation of $A \supset B$ (resp. of C).

- Nothing else in an $\text{NI}^{2\vee}$ -derivation.

The natural deduction system NI^\vee for propositional intuitionistic logic is the restriction of $\text{NI}^{2\vee}$ to the language \mathcal{L}^\vee . The natural deduction system NI^2 for second order

¹In rule schemata we indicate in square brackets the assumptions which can be discharged by rule applications, whereas in derivation schemata we use square brackets only to indicate an arbitrary number of occurrences of a formula, if the formula is in assumption position, or of the whole sub-derivation having the formula in brackets as conclusion. In derivation schemata, we indicate discharge with roman italics letters (possibly with subscripts) placed above the discharged assumptions and near the rule label. Although according to the definition of derivation (which follows strictly the one of [19]), every assumption (both discharged and undischarged) carries a label, to improve readability we will follow the common convention of omitting the labels from undischarged assumptions.

intuitionistic logic (which is isomorphic to Girard-Reynolds *System F*) is the restriction of $\text{NI}^{2\vee}$ to the language \mathcal{L}^2 .

Adopting the terminology of lambda calculus, we will refer to the instances of the following equation schemata as β - and η -equations (we indicate by $\mathcal{D}[B/X]$ the result of substituting the formula B for the variable X in \mathcal{D}):

$\frac{\frac{\frac{[A]}{\mathcal{D}}}{\frac{B}{A \supset B}} \supset I(n) \quad \mathcal{D}' \quad \frac{A}{B} \supset E}{B} =_{\beta}^{\supset} \frac{\mathcal{D}' \quad [A]}{\mathcal{D} \quad B} \quad (\beta \supset)$	$\frac{\frac{\mathcal{D} \quad A \supset B}{B} \supset E \quad \frac{[A]}{A \supset B} \supset I(n)}{A \supset B} =_{\eta}^{\supset} \frac{\mathcal{D}}{A \supset B} \quad (\eta \supset)$ <p>(provided n does not occur in \mathcal{D})</p>
$\frac{\frac{\mathcal{D} \quad A}{\forall X A} \forall I \quad \frac{A[B/X]}{A[B/X]} \forall E}{A[B/X]} =_{\beta}^{\forall} \frac{\mathcal{D}[B/X]}{A[B/X]} \quad (\beta \forall)$	$\frac{\frac{\mathcal{D} \quad \forall X A}{A} \forall E \quad \frac{A}{\forall X A} \forall I}{\forall X A} =_{\eta}^{\forall} \frac{\mathcal{D}}{\forall X A} \quad (\eta \forall)$
$\frac{\frac{\mathcal{D} \quad A_i}{A_1 \vee A_2} \vee I_i \quad \frac{\frac{[A_1]}{\mathcal{D}_1} \quad \frac{[A_2]}{\mathcal{D}_2}}{C} \vee E(n, m)}{C} =_{\beta}^{\vee} \frac{\mathcal{D} \quad [A_i]}{\mathcal{D}_i \quad C} \quad (\beta \vee)$	$\frac{\mathcal{D} \quad \frac{[A]}{A \vee B} \vee I \quad \frac{[B]}{A \vee B} \vee I}{A \vee B} \vee E(n, m) =_{\eta}^{\vee} \frac{\mathcal{D}}{A \vee B} \quad (\eta \vee)$ <p>(provided n, m do not occur in \mathcal{D})</p>

As usual, we will refer to the rewriting rules obtained by orienting these equations from left to right as β -, η -*reductions* and to those obtained by orienting these equations from right to left as β -, η -*expansions*.

The deductive patterns displayed on the left-hand side (respectively right-hand side) of these equations will be referred to as β -, η -*redexes* (resp. *reduct*). A derivation is β -, η -normal if and only if it contains no β -, η -redex.

As is well known (see, e.g., [9], p. 76), in order for normal derivations in NI^{\vee} to enjoy the sub-formula property, a further kind of equations (which we will call γ) need to be assumed (we indicate with $\dagger E$ the application of an elimination rule for “some” connective \dagger and with $\overline{\mathcal{D}}$ the derivations of its minor premises):

$$\frac{\frac{\frac{[A]}{\mathcal{D}_1} \quad \frac{[B]}{\mathcal{D}_2}}{C} \vee E(n, m) \quad \overline{\mathcal{D}}}{D} \dagger E =_{\gamma}^{\vee} \frac{\mathcal{D} \quad \frac{[A]}{\mathcal{D}_1} \quad \overline{\mathcal{D}}}{D} \dagger E \quad \frac{\frac{[B]}{\mathcal{D}_2} \quad \overline{\mathcal{D}}}{D} \dagger E}{D} \vee E(n, m) \quad (\gamma \vee)$$

The left to right orientation of γ -equations are usually called *permutations* (sometimes also *commutations*) rather than reductions. However, we will speak of γ -redexes and γ -normal derivations. We will use *conversion* to cover reductions, expansions and permutations.

In categorial logic (see, e.g. [20]), as well as in the literature on typed lambda calculi with sums (see for instance [12]), however, one usually considers a more general equation schema, namely this:

$$\begin{array}{c}
 \begin{array}{c}
 \mathcal{D} \quad \begin{array}{c} \overset{n}{[A]} \\ \mathcal{D}_1 \end{array} \quad \begin{array}{c} \overset{m}{[B]} \\ \mathcal{D}_2 \end{array} \\
 \hline
 A \vee B \quad C \quad C \quad \vee E(n, m) \\
 \hline
 [C] \\
 \mathcal{D}_3 \\
 D
 \end{array}
 \quad =_{\gamma_g}^{\vee} \quad
 \begin{array}{c}
 \begin{array}{c} \overset{n}{[A]} \\ \mathcal{D}_1 \end{array} \quad \begin{array}{c} \overset{m}{[B]} \\ \mathcal{D}_2 \end{array} \\
 \hline
 [C] \quad [C] \\
 \mathcal{D} \quad \mathcal{D}_3 \quad \mathcal{D}_3 \\
 \hline
 A \vee B \quad D \quad D \quad \vee E(n, m) \\
 \hline
 D
 \end{array}
 \quad (\gamma_g \vee)
 \end{array}$$

of which γ is just an instance obtained by taking \mathcal{D}_3 to consist of the application of $\dagger E$ alone. We call *generalized permutations* the left-to-right orientation of these equations.

We indicate by $=_{\beta\eta\gamma}^{\vee 2\vee}$ the closure under substitution of the equivalence relation induced by these equations on $\text{NI}^{2\vee}$ -derivations (by removing one or more of the subscripts or superscripts we indicate the opportune restrictions of this equivalence).

It is worth observing that the schema $(\gamma_g \vee)$ is more general than $(\gamma \vee)$ in two respects: first, it allows the downwards permutation of an application of $\vee E$ across more than one rule application at once; second, it allows the downward permutations of $\vee E$ not only when its conclusion is the major premise of an elimination rule, but also when it is the premise of an introduction or the minor premise of an elimination. In fact, the equational theory induced by $(\gamma \vee)$ is strictly contained in the one induced $(\gamma_g \vee)$:

Proposition 2.1 ($=_{\gamma}^{\vee} \subseteq =_{\gamma_g}^{\vee}$). *There are derivations \mathcal{D}_1 and \mathcal{D}_2 such that $\mathcal{D}_1 =_{\gamma_g}^{\vee} \mathcal{D}_2$ and $\mathcal{D}_1 \neq_{\gamma}^{\vee} \mathcal{D}_2$.*

Proof. The proposition is established by remarking that the rewriting system consisting of the γ -conversion is strongly normalizing ([14]), while the one consisting of the γ_g -conversion is not. For the latter, consider the following [8, 2, 12]:

$$\begin{array}{c}
 \begin{array}{c}
 \mathcal{D} \quad \begin{array}{c} \overset{n}{[A]} \\ \mathcal{D}' \end{array} \quad \begin{array}{c} \mathcal{D} \quad \mathcal{D}' \end{array} \quad \begin{array}{c} \overset{n}{[B]} \\ \mathcal{D}' \end{array} \\
 \hline
 A \vee B \quad C \quad A \vee B \quad C \quad C \quad \vee E \\
 \hline
 C \quad C \quad \vee E \\
 \hline
 C
 \end{array}
 \quad \xrightarrow{\gamma_g} \quad
 \begin{array}{c}
 \mathcal{D} \quad \begin{array}{c} \overset{n}{[A]} \\ \mathcal{D}' \end{array} \quad \begin{array}{c} \mathcal{D} \quad \mathcal{D}' \end{array} \quad \begin{array}{c} \overset{n}{[B]} \\ \mathcal{D}' \end{array} \\
 \hline
 A \vee B \quad C \quad A \vee B \quad C \quad C \quad \vee E \\
 \hline
 C \quad C \quad \vee E \\
 \hline
 C
 \end{array}
 \quad \xrightarrow{\gamma_g} \quad
 \begin{array}{c}
 \mathcal{D} \quad \begin{array}{c} \overset{n}{[A]} \\ \mathcal{D}' \end{array} \quad \begin{array}{c} \mathcal{D} \quad \mathcal{D}' \end{array} \quad \begin{array}{c} \overset{n}{[B]} \\ \mathcal{D}' \end{array} \\
 \hline
 A \vee B \quad C \quad A \vee B \quad C \quad C \quad \vee E \\
 \hline
 C \quad C \quad \vee E \\
 \hline
 C
 \end{array}
 \quad \xrightarrow{\gamma_g} \quad \dots
 \end{array}$$

□

As the generalized permutation $(\gamma_g \vee)$ is less well-behaved than the standard one, and moreover $\beta\eta\gamma$ -normal derivations already enjoy the subformula property, one may question the choice of replacing $(\gamma \vee)$ with $(\gamma_g \vee)$.

Nonetheless, a strong reason for the adoption of $(\gamma_g \vee)$ comes from the idea to treat two equivalent derivations as representing the same proof. According to Prawitz [14]

proofs should be understood along the lines of intuitionism as the process of mental construction performed by an idealized knowing subject and formal derivations as their linguistic representations. Equivalent derivations thus “denote” the same object (as it happens, for Frege, in the case of ‘The morning star’ and ‘The evening star’).

To appreciate why this prompts the adoption of $(\gamma_g \vee)$ in place of $(\gamma \vee)$, we perform a brief detour to the extension of \mathcal{L}^\vee and \mathbf{NI}^\vee with the \perp and its elimination rule. In this case, the standard permutation for \perp is [9]:

$$\frac{\frac{\mathcal{D}}{\frac{\perp}{C} \perp E} \quad \overline{\mathcal{D}}}{D} \dagger E =_{\gamma} \frac{\mathcal{D}}{\frac{\perp}{D} \perp E} \quad (\gamma \perp)$$

whereas the generalized permutation has the following form:

$$\frac{\frac{\mathcal{D}}{\frac{\perp}{C} \perp E} \quad \mathcal{D}'}{D} =_{\gamma_g} \frac{\mathcal{D}}{\frac{\perp}{D} \perp E} \quad (\gamma_g \perp)$$

Now, according to the so-called Brouwer-Heyting-Kolmogorov explanation, a proof of $A \supset B$ is a function from proofs of A to proofs of B and \perp is defined as what there is no proof of. Thus each formula of the form $\perp \supset C$ can have at most one proof (in fact, the same is true for any formula of the form $C \supset \perp$).

Thus, the two derivations:

$$\frac{\frac{\frac{\perp}{A} \perp E}{A \vee B} \vee I_1}{\perp \supset (A \vee B)} \supset I(n) \quad \frac{\frac{\frac{\perp}{B} \perp E}{A \vee B} \vee I_2}{\perp \supset (A \vee B)} \supset I(n)$$

should represent the same BHK proof of $\perp \supset (A \vee B)$. However, being distinct derivations in $\beta\eta\gamma$ -normal form, they are not $\beta\eta\gamma$ -equivalent.² This shows that $=_{\beta\eta\gamma}$ is too weak an equivalence to properly represent identity of proof. On the other hand, the two derivations are equivalent (and thus, as desired, denote the same proof) if one replaces the standard permutation with the generalized one.

Before introducing the Russell-Prawitz translation in the next section, we remark that we will implicitly assume a further equivalence on derivations (corresponding to α -equivalence on simply typed λ -terms) consisting in equating derivations which differ only in the labels of discharged assumptions (see [19]).

²In general, as a consequence of the Church-Rosser property of the rewriting system consisting of β - and η - reductions and γ -permutations we have that: If two derivations in $\beta\eta\gamma$ -normal form are distinct, then they are not $\beta\eta\gamma$ -equivalent.

3 Properties of the Russell-Prawitz translation

Prawitz [13] showed how to extend Russell's translation of formulas of $\mathcal{L}^{2\vee}$ into formulas of \mathcal{L}^2 into a translation of $\text{NI}^{2\vee}$ into derivations of NI^2 . The Russell-Prawitz translation (for short, RP-translation), is defined as follows:

Definition 3.1 (Russell-Prawitz translation: $\mathcal{L}^{2\vee} \mapsto \mathcal{L}^2$). *The RP-translation of an $\mathcal{L}^{2\vee}$ -formula A is the \mathcal{L}^2 -formula A^* is defined by induction on the number of logical signs in A as follows:*

$$\begin{aligned} Y^* &\equiv Y \\ (A \supset B)^* &\equiv A^* \supset B^* \\ (\forall Y A)^* &\equiv \forall Y A^* \\ (A \vee B)^* &\equiv \forall X((A^* \supset X) \supset (B^* \supset X) \supset X) \end{aligned}$$

Definition 3.2 (Russell-Prawitz translation: $\text{NI}^{2\vee} \mapsto \text{NI}^2$). *The RP-translation of an $\text{NI}^{2\vee}$ -derivation \mathcal{D} is the NI^2 -derivation \mathcal{D}^* defined by induction on the number of inference rules applied in \mathcal{D} as follows:*

- if $\mathcal{D} \equiv \overset{n}{A}$, then $\mathcal{D}^* \equiv \overset{n}{A^*}$;
- if $\mathcal{D} \equiv \frac{\mathcal{D}'}{A \vee B} \vee I_1'$, then $\mathcal{D}^* \equiv \frac{\frac{\frac{A^* \supset X}{X} \supset I}{(B^* \supset X) \supset X} \supset I}{(A^* \supset X) \supset (B^* \supset X) \supset X} \supset I(n) \vee I$ (provided n does not occur in \mathcal{D}'^*);
- if $\mathcal{D} \equiv \frac{\mathcal{D}'}{B} \vee I_2'$, then $\mathcal{D}^* \equiv \frac{\frac{\frac{B^* \supset X}{X} \supset I}{(B^* \supset X) \supset X} \supset I(n)}{(A^* \supset X) \supset (B^* \supset X) \supset X} \supset I \vee I$ (provided n does not occur in \mathcal{D}'^*);
- if $\mathcal{D} \equiv \frac{\overset{n}{[A]} \quad \overset{m}{[B]}}{A \vee B \quad C \quad C} \vee E(n, m)$, then

$$\mathcal{D}^* \equiv \frac{\frac{\mathcal{D}'^*}{(A^* \supset C^*) \supset (B^* \supset C^*) \supset C^*} \vee E \quad \frac{\frac{\overset{n}{[A^*]}}{\mathcal{D}_1^*} C^*}{A^* \supset C^*} \supset I(n)}{(B^* \supset C^*) \supset C^*} \supset E \quad \frac{\frac{\overset{m}{[B^*]}}{\mathcal{D}_2^*} C^*}{B^* \supset C^*} \supset I(m)}{C^*} \supset E$$

- all other rules are translated in a trivial way.

The RP-translation maps β -equivalent derivations into β -equivalent derivations in NI^2 :

Proposition 3.1 ($=_{\beta}^{2\supset\vee} \mapsto =_{\beta}^{2\supset}$). *If $\mathcal{D}_1 =_{\beta}^{2\supset\vee} \mathcal{D}_2$ then $\mathcal{D}_1^* =_{\beta}^{2\supset} \mathcal{D}_2^*$.*

Proof. It suffices to show this in the case of a \vee -redex. If the redex contains an application of $\vee I_1$, then we just verify that:

$$\begin{array}{c}
 \frac{\frac{A^* \supset X \quad \mathcal{D}^*}{X} \supset E}{(B^* \supset X) \supset X} \supset I \\
 \frac{(A^* \supset X) \supset (B^* \supset X) \supset X}{(A \vee B)^*} \supset I (m) \\
 \frac{(A \vee B)^*}{(A^* \supset C^*) \supset (B^* \supset C^*) \supset C^*} \vee E \\
 \frac{(A^* \supset C^*) \supset (B^* \supset C^*) \supset C^*}{(B^* \supset C^*) \supset C^*} \supset E \\
 \frac{(B^* \supset C^*) \supset C^*}{C^*} \supset E
 \end{array}
 \quad
 \begin{array}{c}
 \frac{n_1}{[A^*]} \\
 \mathcal{D}_1^* \\
 C^* \\
 \frac{C^*}{A^* \supset C^*} \supset I (n_1)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{n_2}{[B^*]} \\
 \mathcal{D}_2^* \\
 C^* \\
 \frac{C^*}{B^* \supset C^*} \supset I (n_2)
 \end{array}
 \quad
 \begin{array}{c}
 \mathcal{D}^* \\
 [A^*] \\
 \mathcal{D}_1^* \\
 C^*
 \end{array}
 \quad
 =_{\beta}$$

The case where the redex contains an application of $\vee I_2$ is treated similarly. \square

Analogous results do not hold for the equivalence relations η and γ (see e.g. [9], p. 85): in particular, it is not the case that whenever $\mathcal{D}_1 =_{\eta}^{\vee} \mathcal{D}_2$, then $\mathcal{D}_1^* =_{\eta}^{2\supset} \mathcal{D}_2^*$, actually not even $\mathcal{D}_1^* =_{\beta\eta}^{2\supset} \mathcal{D}_2^*$. Similarly, it is not the case that whenever $\mathcal{D}_1 =_{\gamma}^{\vee} \mathcal{D}_2$, then $\mathcal{D}_1^* =_{\beta\gamma}^{2\supset} \mathcal{D}_2^*$. To see this, it is enough to consider any instance of $(\eta\vee)$ and $(\gamma_g\vee)$ in which the derivation \mathcal{D} of the major premise of the application of $\vee E$ constituting the redex consists of the sole assumption of $A \vee B$.

In spite of this, it is possible to show that particular classes of instances of $(\eta\vee)$ and $(\gamma_g\vee)$ are preserved by the RP-translation, in particular, those instances in which the derivation of the major premise of the application of $\vee E$ displayed in the equation schemata is closed (we will refer to such instances as *m-closed*). The equivalence relations induced by these instances of $(\eta\vee)$ and $(\gamma_g\vee)$ will be indicated with $=_{\eta}^{\vee mc}$ and $=_{\gamma_g}^{\vee mc}$.

To prove this fact we rely on a restricted form of normalization for $\text{NI}^{2\vee}$ (namely, that any derivation \mathcal{D} can be β -reduced to a β -normal derivation $\tilde{\mathcal{D}}$) and on the following:

Proposition 3.2. *All closed β -normal $\text{NI}^{2\vee}$ -derivations ends with an application of an introduction rule.*

Proof. The proof is by induction on the number of applications of elimination rules in a closed β -normal derivation \mathcal{D} :

- if \mathcal{D} has no elimination rules, then, as it cannot consist of an assumption (otherwise it would be open), it must end with an introduction.
- Suppose for *reductio* that \mathcal{D} ends with an application of an elimination rule. As \mathcal{D} is closed and β -normal, so is the derivation \mathcal{D}' of the (major) premise of the application of the elimination rule with which \mathcal{D} is supposed to end. As \mathcal{D}' has one elimination rule less than \mathcal{D} , we can apply the induction hypothesis: \mathcal{D}' ends with an introduction rule. Hence, \mathcal{D} is not normal, in contradiction with the assumption that it is. Thus, \mathcal{D} cannot end with an application of an elimination rule.

□

A consequence of this is that the m-closed instances of $(\gamma_g \vee)$ and $(\eta \vee)$ are contained in the equational theory $=_{\beta}^{2\supset \vee}$:

Proposition 3.3. *If $\mathcal{D}' =_{\gamma_g^{mc}}^{\vee} \mathcal{D}''$ then $\mathcal{D}' =_{\beta}^{2\supset \vee} \mathcal{D}''$*

Proof. Consider an instance of $(\gamma_g \vee)$ in which the derivation \mathcal{D} of $A \vee B$ is *closed*. Call \mathcal{D}' and \mathcal{D}'' the left-hand side and right-hand side of this instance of $(\gamma_g \vee)$. We show that $\mathcal{D}' =_{\beta}^{2\supset \vee} \mathcal{D}''$.

Since \mathcal{D} is closed, it can be β -reduced into a β -normal derivation $\mathcal{D}^{\#}$ which (by proposition 3.2) consists of a derivation $\mathcal{D}^{\#}$ of either A or B to which an application of one of the introduction rules is appended. If we assume that the rule applied is $\vee I_1$ (the alternative case is similar), the two members of the γ_g -equation β -reduce (respectively) to the following derivations:

$$\begin{array}{c}
 \frac{\mathcal{D}^{\#'} \quad \frac{A}{A \vee B} \vee I_1}{[C]} \quad \frac{\frac{[A]}{\mathcal{D}_1} \quad \frac{[B]}{\mathcal{D}_2}}{C} \vee E(n, m) \\
 \mathcal{D}_3 \\
 D
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\mathcal{D}^{\#'} \quad \frac{A}{A \vee B} \vee I_1}{D} \quad \frac{\frac{[A]}{\mathcal{D}_1} \quad \frac{[B]}{\mathcal{D}_2}}{[C]} \vee E(n, m) \\
 \mathcal{D}_3 \\
 D
 \end{array}$$

which are clearly β -equivalent.

□

Proposition 3.4. *If $\mathcal{D}' =_{\eta^{mc}}^{\vee} \mathcal{D}''$ then $\mathcal{D}' =_{\beta}^{2\supset \vee} \mathcal{D}''$*

Proof. Consider an instance of $(\eta \vee)$ in which \mathcal{D} is *closed*. Call \mathcal{D}' and \mathcal{D}'' the left-hand side and right-hand side of this instance of $(\eta \vee)$. We show $\mathcal{D}' =_{\beta}^{2\supset \vee} \mathcal{D}''$.

As in the proof of the previous proposition, the two members of the η -equation β -reduce (respectively) to the following derivations:

$$\begin{array}{c}
 \frac{\mathcal{D}^{\#'} \quad \frac{A}{A \vee B} \vee I_1}{A \vee B} \vee I_1 \quad \frac{\frac{A}{A \vee B} \vee I_1 \quad \frac{B}{A \vee B} \vee I_1}{A \vee B} \vee E(n, m) \\
 \frac{\mathcal{D}^{\#'} \quad \frac{A}{A \vee B} \vee I_1}{A \vee B} \vee I_1
 \end{array}$$

which are clearly β -equivalent. \square

Remark 3.1. Propositions analogous to 3.3 and 3.4 hold for the m-closed instances of $(\eta\forall)$ and $(\eta\supset)$ as well.

As by Propositions 3.3 and 3.4 the m-closed instances of $(\gamma_g\forall)$ and of $(\eta\forall)$ are included in the equational theory $=_{\beta}^{2\supset\forall}$, and moreover by Proposition 3.1 the RP-translation maps $\beta^{2\supset\forall}$ -equivalent derivations in $\text{NI}^{2\forall}$ into $\beta^{2\supset}$ -equivalent derivations in NI^2 , we have the following:

Corollary 3.5 $(=_{\gamma_g^{mc}}^{\forall} \xrightarrow{*} =_{\beta}^{2\supset})$. If $\mathcal{D}_1 =_{\gamma_g^{mc}}^{2\supset\forall} \mathcal{D}_2$ then $\mathcal{D}_1^* =_{\beta}^{2\supset} \mathcal{D}_2^*$.

Corollary 3.6 $(=_{\eta^{mc}}^{\forall} \xrightarrow{*} =_{\beta}^{2\supset})$. If $\mathcal{D}_1 =_{\eta^{mc}}^{2\supset\forall} \mathcal{D}_2$ then $\mathcal{D}_1^* =_{\beta}^{2\supset} \mathcal{D}_2^*$.

Remark 3.2. A proposition analogous to 3.3 and 3.4 and a corollary analogous to 3.5 and 3.6 can be established for the so-called simplification conversions [14, §II.3.3.2.1], which are the left-to-right orientations of the instances of the following equation schema:

$$\frac{\begin{array}{ccc} \mathcal{D} & \mathcal{D}_1 & \mathcal{D}_2 \\ A_1 \vee A_2 & C & C \end{array}}{C} =_{\sigma} \mathcal{D}_i \quad (\sigma\vee)$$

provided the displayed application of $\vee E$ discharges no assumption in \mathcal{D}_i

In the next sections we will show that the corollaries 3.6 and 3.5 are instances of a more general phenomenon, namely the *naturality* of natural deduction derivations.

4 The naturality of NI^2 -derivations

In this section we introduce a naturality condition for NI^2 -derivations, well-known from categorial approaches to logic, in purely proof-theoretic terms.

The presentation will rely in an essential way on the notion of substitution of a formula A for a variable X both within a formula C , $C[A/X]$, and within a derivation \mathcal{D} , $\mathcal{D}[A/X]$. For simplicity, we identify formulas and derivations which can be obtained from each other by renaming the bound variables. Hence,

Remark 4.1. We assume the following:

- i. Substitution is always defined, though, in some cases, the substitution $(\forall YF)[A/X] = \forall Y'(F'[A/X])$ might require a renaming of the bound variables of F .
- ii. Given two derivations \mathcal{D} and \mathcal{D}' , no bound variables in \mathcal{D} occurs free in \mathcal{D}' .

These assumptions will be needed in the proof of the main theorem of this section. This, as well as most results in this section below will be shown for particular classes of derivation which are defined relative to the choice of a particular variable:

Definition 4.1 (X-safety). An NI^2 -derivation \mathcal{D} is X-safe, if, for all application $\frac{\forall YA}{A[B/Y]} \forall E$ of $\forall E$ in \mathcal{D} , X does not occur in B .

Remark 4.2. Observe that X-safety is preserved by β -reduction, i.e. if \mathcal{D} β -reduces to \mathcal{D}' and \mathcal{D} is X-safe, so is \mathcal{D}' .

The identification of formulas and derivations up to renaming the bound variables has also the following consequence (which will be exploited in section 5):

Proposition 4.1. For any NI^{2V} -derivation \mathcal{D} ending with an application of $\forall E$, there is always a variable X such that in the RP-translation \mathcal{D}^* of \mathcal{D} (modulo α -equivalence):

1. the premise of the last application of $\forall E$ (i.e. the one used in translating the application of $\forall E$ with which \mathcal{D} ends) is $\forall X((A \supset X) \supset (B \supset X) \supset X)$;
2. \mathcal{D}^* is X-safe

Proof. Choose an X that does not occur in \mathcal{D} . By induction on \mathcal{D} one shows that \mathcal{D}^* does not contain free occurrences of X . Thus if X occurs at all in \mathcal{D}^* it occurs bounded in a formula of the form $\forall X((A^* \supset X) \supset (B^* \supset X) \supset X)$. Now, for any occurrence of an elimination rule $\frac{\forall ZA}{A[B/Z]} \forall E$, all occurrences of X in B must be bound, and it suffices to rename them. \square

In this section, from now on when we speak of derivations we mean NI^2 -derivations. Besides using substitution, we will need a further notion in order to present naturality in proof-theoretic terms: when C is a formula of a particular form, X is a variable and \mathcal{D} is a derivation of B from (undischarged) assumptions A, Δ , the C -expansion of \mathcal{D} relative to X , will be defined as a particular derivation, to be indicated with $C \parallel \mathcal{D} \parallel_X$, of $C[B/X]$ from $C[A/X]$ and Δ .³

In section 4.1 we will introduce the particular form of formulas we will use in the definition of C -expansion. We will devote section 4.2 to the latter notion and its properties. Finally we will introduce the naturality condition in section 4.3.

4.1 Sp-X formulas and derivations

In this subsection we make precise which is the particular form of the formulas mentioned in the preceding informal remarks and prove some results about derivations whose assumptions and conclusions are formulas of this form.

Definition 4.2 (strictly positive formulas and derivations). A formula C is strictly positive relative to X (abbreviated *sp-X*) iff:

- $C \equiv Z$ (where, possibly, $Z = X$);
- $C \equiv A \supset B$, provided X does not occur in A and B is *sp-X*;
- $C \equiv \forall YA$, provided A is *sp-X* and $Y \neq X$.

³Actually, the restriction to formulas of a particular form could be avoided, but for our goals we consider only this simplified case. See also remark 4.6 below.

A derivation \mathcal{D} of C from Γ is *sp-X* iff all formulas in Γ and C are *sp-X*.

Remark 4.3. Let $\forall \bar{Y}_i$ (for $1 \leq i \leq n$) denote finite (possibly empty) sequences of quantifiers whose variables are distinct from X . If C is *sp-X* then $C \equiv \forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset Z) \dots))$ where X does not occur in any of the F_i (for $1 \leq i \leq n$).

In order to prove the main theorem of this section we first need to show that in a β -normal, X -safe, *sp-X* derivation only *sp-X* formulas occur. Let's define the notion of sub-formula as follows:

Definition 4.3 (Sub-formula). *The sub-formulas of A are defined by induction on the number of logical signs in A as follows:*

- if $A \equiv X$, the only sub-formula of A is A itself;
- if $A \equiv B \supset C$, the sub-formulas of A are A itself and the sub-formulas of B and C ;
- if $A \equiv \forall XB$, the sub-formulas of A are A itself and the sub-formulas of B ;

Clearly, the following hold:

Lemma 4.2. *If A is *sp-X*, so are all its sub-formulas.*

Proof. By induction on the number of logical signs in A . □

Lemma 4.2 is not enough to warrant that in a β -normal *sp-X* derivation all formulas are *sp-X*, since normal derivations in NI^2 do not enjoy the sub-formula property. However, we can show that if a *sp-X* derivation is X -safe, then it enjoys a weakened form of the sub-formula property that is enough to yield the desired result.

Definition 4.4 (X -equivalence). *We say that $A <'_X B$ iff for some $Y \neq X$ and for some C , such that X does not occur in C , $B \equiv A[C/Y]$.*

Let the relation $A <_X B$ and $='_X$ be defined (respectively) as the transitive closure and the reflexive and symmetric closure of the relation $A <'_X B$.

Finally, let the relation $=_X$ be the union of $A <_X B$ and $='_X$ (i.e. the reflexive symmetric and transitive closure of $A <'_X B$).

To establish the weakened sub-formula property for *sp-X* derivations we first prove the following:

Lemma 4.3. *If $A <_X B \supset C$, then there exist sub-formulas A_1 and A_2 of A such that $A_1 \leq_X B$ and $A_2 \leq_X C$;*

Proof. We prove the two parts of the lemma separately: Let $A <_X B \supset C$. Then there exists a finite sequence of substitutions (whose image is made of formulas not containing X) $\theta_1, \dots, \theta_n$ such that $A\theta_1 \dots \theta_n \equiv B \supset C$. If A is of the form $A_1 \supset A_2$, then $A\theta_1 \dots \theta_n \equiv A_1\theta_1 \dots \theta_n \supset A_2\theta_1 \dots \theta_n$, which proves the claim. Otherwise, since A cannot be of the form $\forall ZA'$ (as $(\forall ZA')\theta_1 \dots \theta_n \equiv \forall Z(A\theta'_1 \dots \theta'_n)$, where θ'_i is obtained by renaming all occurrences of Y), A must be a variable Y , and then $A[B/Y] \equiv B$ and $A[C/Y] \equiv C$, so we can take $A_1 \equiv A_2 \equiv A$. □

Lemma 4.4. *If $A =_X B$ and A is sp-X , then B is sp-X .*

Proof. By induction on A one shows that, if C does not contain occurrences of X , then A is sp-X if and only if $A[C/Y]$ (for $Y \neq X$) is sp-X . This proves the claim for $='_X$ (the reflexive and symmetric closure of $<'_X$). The claim can then be extended to $=_X$ by induction to the application to A of a finite number of substitutions. \square

We can now establish the following weakened form of the sub-formula property for β -normal, X -safe and sp-X derivations:

Proposition 4.5. *Let \mathcal{D} be a β -normal, X -safe and sp-X derivation. Then, for any formula F occurring in \mathcal{D} ,*

- i. for some formula C , which is a sub-formula either of the conclusion of \mathcal{D} , or of some undischarged assumption of \mathcal{D} , $F =_X C$;*
- ii. moreover, if \mathcal{D} ends by an elimination rule, then it has a principal branch, i.e. a sequence of sp-X formulas A_0, \dots, A_n such that*
 - A_0 is an undischarged assumption of \mathcal{D} ;*
 - A_n is the conclusion of \mathcal{D} ;*
 - for all $1 \leq i \leq n - 1$, A_i is the major premise of an elimination rule whose consequence is A_{i+1} .*

Proof. We argue by induction on \mathcal{D} . If \mathcal{D} consists solely of an assumption, there is nothing to prove. If \mathcal{D} ends with an application of an introduction rule, we must consider two cases:

1. \mathcal{D} ends with an application of $\supset\text{I}$:

$$\frac{\begin{array}{c} [A] \\ \mathcal{D}' \\ B \end{array}}{A \supset B} \supset\text{I} (n)$$

Let F be a formula occurring in \mathcal{D} . Unless F is $A \supset B$, in which case there is nothing to prove, F occurs in \mathcal{D}' . First observe that, as \mathcal{D} is sp-X , A does not contain occurrences of X , hence \mathcal{D}' is sp-X too. Then, by induction hypothesis, two possibilities arise: either for some sub-formula C of an undischarged assumption C' of \mathcal{D}' , $F =_X C$; or for some sub-formula C of B , $F =_X C$. In the first case, if C' is different from A , then it is an undischarged assumption of \mathcal{D} and we are done; if C' is A , we conclude by remarking that A is a sub-formula of the conclusion $A \supset B$ of \mathcal{D} . In the second case, we conclude similarly by remarking that B is a sub-formula of the conclusion $A \supset B$.

2. \mathcal{D} ends with an application of $\forall\text{I}$ rule:

$$\frac{\begin{array}{c} \mathcal{D}' \\ A \end{array}}{\forall YA} \forall\text{I}$$

Let F be a formula occurring in \mathcal{D} . Again, unless F is $\forall YA$, in which case there is nothing to prove, F occurs in \mathcal{D}' . Then, by induction hypothesis, either for some sub-formula C of an undischarged assumption C' of \mathcal{D}' , $F =_X C$, or for some sub-formula C of A , $F =_X C$. In the first case, we are done, as all undischarged assumptions of \mathcal{D}' are undischarged assumptions of \mathcal{D} ; in the second case, we conclude by remarking that A is a sub-formula of the conclusion $\forall YA$ of \mathcal{D} .

If \mathcal{D} ends with an application of an elimination rule, again we must consider two cases:

1. \mathcal{D} ends with an application of $\supset E$:

$$\frac{\frac{\mathcal{D}_1}{A \supset B} \quad \frac{\mathcal{D}_2}{A}}{B} \supset E$$

As \mathcal{D} is β -normal, by proposition 3.2 the subderivation \mathcal{D}_1 cannot end with an introduction rule. Hence, by induction hypothesis, there exists a principal branch $A_0, \dots, A_n \equiv A \supset B$ in \mathcal{D}_1 . It can be easily shown by induction that, for all $1 \leq i \leq n$, there exists a sub-formula A' of A_0 such that $A' <_X A_i$. Hence, in particular, there exists a sub-formula A' of A_0 such that $A' <_X A \supset B$.

Now, if F occurs in \mathcal{D} then, unless F is B , in which case we are done, F must occur either in \mathcal{D}_1 , either in \mathcal{D}_2 . In the first case, by induction hypothesis, either for some C sub-formula of an undischarged assumption of \mathcal{D}_1 , $F =_X C$, in which case we are done, or for some sub-formula C of $A \supset B$, $F =_X C$, in which case we use the fact that $A \supset B =_X A'$ and the transitivity of $=_X$. In the second case, either for some C sub-formula of an undischarged assumption of \mathcal{D}_2 , $F =_X C$, in which case we are done, or for some sub-formula of A , $F =_X C$; in this last case, as A is a sub-formula of $A \supset B$, by lemma 4.3, there exists a sub-formula A'' of A' (and hence of A_0), such that $A'' <_X A$.

Finally, take $A_{n+1} \equiv B$ and we obtain a principal branch for \mathcal{D} , since, by lemma 4.2, A_{n+1} is sp-X .

2. \mathcal{D} ends with an application of $\forall E$:

$$\frac{\frac{\mathcal{D}'}{\forall ZA}}{A[B/Z]} \forall E$$

As \mathcal{D} is safe, X does not occur in B . Moreover, as \mathcal{D} is β -normal, by proposition 3.2 the subderivation \mathcal{D}' cannot end by an introduction. Hence, by induction hypothesis, there exists a principal branch $A_0, \dots, A_n = \forall ZA$ in \mathcal{D}' . Then, there exists a sub-formula A' of A_0 such that $A' <_X \forall ZA$.

Now, if F occurs in \mathcal{D} then, unless $F \equiv A[B/Z]$, in which case we are done, F must occur in \mathcal{D}' . Then, by induction hypothesis, either there exists a sub-formula C of an undischarged assumption of \mathcal{D}' such that $F =_X C$, in which case we are done, or for

some sub-formula C of $\forall Z A$, $F =_X C$, in which case we use the fact that $A' =_X \forall Z A$ as well as the transitivity of $=_X$.

Finally, take $A_{n+1} \equiv A[B/Z]$ and we obtain a principal branch for \mathcal{D} , since, by lemma 4.4 A_{n+1} is sp- X .

□

As a corollary of proposition 4.5 we finally obtain the following:

Corollary 4.6. If \mathcal{D} is X -safe, β -normal and sp- X , all formulas occurring in \mathcal{D} are sp- X .

Proof. By proposition 4.5, if a formula F occurs in \mathcal{D} , then for some formula C , which is either a sub-formula of an undischarged assumption of \mathcal{D} , either a sub-formula of the conclusion of \mathcal{D} , $F =_X C$. By lemma 4.2, C is sp- X . Hence, by lemma 4.4, F must be sp- X . □

4.2 The C -expansion of a derivation relative to X

We introduce the following:

Notational convention 4.1. For ease of notation, whenever the context makes clear which is the variable on which we perform some substitutions and the variable relative to which we C -expand a derivation, we will write $\mathcal{D}[A]$, $C[A]$ and $C\|\mathcal{D}\|$ for $\mathcal{D}[A/X]$, $C[A/X]$ and $C\|\mathcal{D}\|_X$ respectively. To enhance readability, we will sometimes colour the notation for the substitution of A for X in C and for the C -expansions of \mathcal{D} relative to X writing $C[A]$ and $C\|\mathcal{D}\|$.

Remark 4.4. The following basic facts about substitution:

- i. If X does not occur in C , then $C[A/X] = C$ for all A ;
- ii. For all A, X and C , $(\forall X C)[A/X] = \forall X C$;
- iii. If X does not occur in C , nor Y in A , then $F[C/Y][A/X] = F[A/X][C/Y]$ for all F .

will be therefore expressed as follows:

- ia. If X does not occur in C , then for all A $C[A] = C$;
- ii. For all A, X and C , $\forall X C[A] = \forall X C$;
- iiia. If X does not occur in C , nor Y in A , then for all F $F[C/Z][A] = F[A][C/Z]$.

We now define the notion which together with substitution allows the formulation of the naturality condition:

Definition 4.5 (C -expansion of \mathcal{D} relative to X). If C is sp- X and \mathcal{D} is a derivation of B from undischarged assumptions A, Δ , we call the C -expansion of \mathcal{D} relative to X , notation $C\|\mathcal{D}\|_X$, the derivation of $C[B/X]$ from $C[A/X], \Delta$ defined as follows (using the notational convention

4.1 and leaving the assumptions in Δ implicit, we thus have that $C\|\mathcal{D}\|_X \equiv \begin{matrix} C[A] \\ C\|\mathcal{D}\| \\ C[B] \end{matrix}$ which we

shorten further to $\begin{matrix} C[A] \\ C\|\mathcal{D}\| \\ C[B] \end{matrix}$):

- A. If X does not occur in C , then $C \llbracket \mathcal{D} \rrbracket$ just consists of the assumption of C .
 B. Otherwise $C \llbracket \mathcal{D} \rrbracket$ is defined by induction on C :
- i. If $C \equiv X$ then $C \llbracket \mathcal{D} \rrbracket \equiv \mathcal{D}$;
 - ii. If $C \equiv \forall Y F$ then, as C is sp - X , F is sp - X too, we define

$$\forall Y F \llbracket \mathcal{D} \rrbracket \equiv \frac{\frac{\forall Y F \llbracket A \rrbracket}{F \llbracket \begin{smallmatrix} A \\ \mathcal{D} \\ B \end{smallmatrix} \rrbracket} \forall E}{\forall Y F \llbracket B \rrbracket} \forall I \quad \equiv \quad \frac{\frac{\forall Y F \llbracket A \rrbracket}{F \llbracket \begin{smallmatrix} A \\ \mathcal{D} \\ B \end{smallmatrix} \rrbracket} \forall E}{\forall Y F \llbracket B \rrbracket} \forall I$$

(where the substitution $(\forall Y F)[A/X] = \forall Y'(F'[A/X])$ might require a renaming of the bound variables of F , cf. remark 4.1i. above).

- iii. If $C \equiv F \supset G$ then, since C is sp - X , X does not occur in F . Thus $F \llbracket A \rrbracket \equiv F \llbracket B \rrbracket \equiv F$ and $F \supset G \llbracket A \rrbracket \equiv F \supset G \llbracket A \rrbracket$ and $F \supset G \llbracket B \rrbracket \equiv F \supset G \llbracket B \rrbracket$. Moreover, G is sp - X . We can then define

$$C \llbracket \mathcal{D} \rrbracket \equiv \frac{\frac{F \supset G \llbracket A \rrbracket}{G \llbracket \begin{smallmatrix} A \\ \mathcal{D} \\ B \end{smallmatrix} \rrbracket} \supset E}{F \supset G \llbracket B \rrbracket} \supset I(n) \quad \equiv \quad \frac{\frac{F \supset G \llbracket A \rrbracket}{F \llbracket A \rrbracket} \supset E}{F \supset G \llbracket B \rrbracket} \supset I(n)$$

Remark 4.5. We observe (for later use in section 6) that, in case X does occur in C (i.e. as the rightmost variable of C) and \mathcal{D} consists just of the assumption of the variable X , i.e. $A \equiv B \equiv X$, the C -expansion of \mathcal{D} relative to X is just the η -long normal form of the derivation consisting of the assumption of C . Thus, given that, in general, the form of an sp - X formula C is $\forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots))$, we schematically represent the C -expansion of \mathcal{D} as follows (with $\frac{\forall \bar{Y}_i G}{G} \forall E$ we indicate (possibly empty) sequences of applications of $\forall E$, and with \mathcal{E}_C^e and \mathcal{E}_C^i we indicate the parts of the expansion constituted by applications of elimination and introduction rules respectively):

$$\begin{array}{c}
\frac{\frac{C}{F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)} \forall E \quad \frac{m_1}{F_1}}{\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)} \supset E \\
\vdots \\
\frac{\frac{\frac{\forall \bar{Y}_n (F_n \supset X)}{F_n \supset X} \forall E \quad \frac{m_n}{F_n}}{\frac{X}{F_n \supset X} \supset I (m_n)} \supset E \quad \frac{m_n}{F_n} \supset E}{\frac{\frac{X}{F_n \supset X} \supset I (m_n)}{\forall \bar{Y}_n (F_n \supset X)} \forall I} \supset E \\
\vdots \\
\frac{\frac{F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)}{C} \supset I (m_1)}{C} \forall I
\end{array}
\quad \equiv \quad
\begin{array}{c}
C \quad \frac{m_1}{F_1} \dots \frac{m_n}{F_n} \\
\mathcal{E}_C^e \\
X \\
\mathcal{E}_C^i (m_1, \dots, m_n) \\
C
\end{array}$$

With \mathcal{E}_C^e and \mathcal{E}_C^i we will likewise indicate the two halves of the η -long normal form of a derivation \mathcal{D} consisting just of the assumption of C :

In the proof of the main theorem in section 4.3 we will need the following:

Lemma 4.7. *If \mathcal{D} is a derivation of B from A, Δ and X does not occur in F , then for all Y not occurring free in \mathcal{D} , the result of substituting F for Y in the C -expansion of \mathcal{D} (relative to X) is equal to the $C[F/Y]$ -expansion of \mathcal{D} (relative to X):*

$$C[F/Y] \llbracket \mathcal{D} \rrbracket \equiv C \llbracket \mathcal{D} \rrbracket [F/Y]$$

Proof. By induction on C , by observing that, as X does not occur in F , the derivation $F \llbracket \mathcal{D} \rrbracket$ consists solely of the assumption of F . \square

Finally we introduce the following:

Notational convention 4.2. If Γ is the multiset of formulas C_1, \dots, C_n , then by $\Gamma[A/X]$ we indicate the multiset of formulas $C_1[A/X] \dots C_n[A/X]$. When X is clear from the context we abbreviate $\Gamma[A/X]$ to $\Gamma[A]$ or, coloured, to $\llbracket \Gamma[A] \rrbracket$. Likewise, if Γ is a multiset of sp- X formulas C_1, \dots, C_n , then by $\llbracket \Gamma \rrbracket$ we indicate the multiset of $\llbracket C_1 \rrbracket \dots \llbracket C_n \rrbracket$.

4.3 The naturality condition

The core insight from category theory is the following: the system NI^2 can be seen as a syntactic category, whose objects are \mathcal{L}^2 formulas and whose morphisms are NI^2 derivations. Then, given a variable X we can associate to any sp- X formula C a functor, whose application to a formula A gives $\llbracket C[A] \rrbracket$ as value, and whose application to a derivation \mathcal{D} gives $\llbracket C \rrbracket \llbracket \mathcal{D} \rrbracket$ as value. Moreover, given an sp- X , X -safe derivation \mathcal{D} of C from Γ , the operation which associates to any formula A the derivation $\mathcal{D}[A]$ yields a family of morphisms θ_A from $\llbracket \Gamma[A] \rrbracket$ to $\llbracket C[A] \rrbracket$. In the terminology of category theory, such a family of morphism is a natural transformation between the functors associated

to Γ and C , provided that for any derivation \mathscr{D}' of B from A, Δ the following diagram commutes:

$$\begin{array}{ccc}
 \Gamma[A] & \xrightarrow{\mathcal{D}[A]} & C[A] \\
 \downarrow \Gamma[\mathcal{D}'] & & \downarrow C[\mathcal{D}'] \\
 \Gamma[B] & \xrightarrow{\mathcal{D}[B]} & C[B]
 \end{array}$$

Using the notions so far introduced, this can be expressed as follows:

Definition 4.6 (naturality condition). Let \mathcal{D} be a sp-X derivation of C from Γ . We say that \mathcal{D} is X-natural iff for any derivation \mathcal{D}' of B from A, Δ , the composition of $\mathcal{D} \llbracket A \rrbracket$ with $\llbracket C \rrbracket \mathcal{D}'$ is β -equal to the composition of $\llbracket \Gamma \rrbracket \mathcal{D}'$ with $\mathcal{D} \llbracket B \rrbracket$, that is iff the following holds:

$$\begin{array}{c} \Gamma \llbracket A \rrbracket \\ \mathcal{D} \llbracket A \rrbracket \\ \mathsf{C} \llbracket A \rrbracket \\ \mathsf{C} \llbracket \mathcal{D}' \rrbracket \\ \mathsf{C} \llbracket B \rrbracket \end{array} \equiv \begin{array}{c} \Gamma \llbracket A \rrbracket \\ \mathcal{D} \llbracket A \rrbracket \\ \mathsf{C} \llbracket \begin{array}{c} A \\ \mathcal{D}' \\ B \end{array} \rrbracket \end{array} \stackrel{2\triangleright}{=}_{\beta} \begin{array}{c} \Gamma \llbracket \begin{array}{c} A \\ \mathcal{D}' \\ B \end{array} \rrbracket \\ \mathcal{D} \llbracket B \rrbracket \\ \mathsf{C} \llbracket B \rrbracket \end{array} \equiv \begin{array}{c} \Gamma \llbracket A \rrbracket \\ \Gamma \llbracket \mathcal{D}' \rrbracket \\ \Gamma \llbracket B \rrbracket \\ \mathcal{D} \llbracket B \rrbracket \\ \mathsf{C} \llbracket B \rrbracket \end{array}$$

Theorem 4.8. *If \mathcal{D} is an X -safe, sp - X derivation, then \mathcal{D} is X -natural.*

Proof. We first prove the result for a normal derivation \mathcal{D} by induction on the number of inference rules applied in \mathcal{D} :

- if \mathcal{D} consists of the assumption of an sp-X formula C then, for all derivations \mathcal{D}' of B from A , we obviously have

$$C \left[\begin{array}{c} A \\ \mathcal{D}' \\ B \end{array} \right] \stackrel{2\supset}{=} C \left[\begin{array}{c} A \\ \mathcal{D}' \\ B \end{array} \right]$$

- if $\mathcal{D} \equiv \frac{\frac{\Gamma}{\mathcal{D}_1} \quad G}{\forall YG} \forall I$ then, for all derivations \mathcal{D}' of B from A , supposing by remark 4.1ii. that no bound variable of \mathcal{D} occurs free in \mathcal{D}' , we have that

- if $\mathcal{D} \equiv \frac{\frac{\Gamma}{\mathcal{D}_1} \quad \forall YG}{G[F/Y]} \forall E$, X does not occur in F (since \mathcal{D} is X -safe). By remark 4.1ii., we can suppose that Y does not occur free in \mathcal{D}' and thus, by lemma 4.7, for all \mathcal{D}' of B from A , $G[F/Y] \parallel \mathcal{D}' \parallel \equiv G \parallel \mathcal{D}' \parallel [F/Y]$. Hence

$$\begin{array}{c}
\frac{\frac{\frac{\Gamma \llbracket A \rrbracket}{\mathcal{D}_1 \llbracket A \rrbracket} \quad \forall Y G \llbracket A \rrbracket}{G \llbracket \begin{array}{c} A \\ \mathcal{D}' \\ B \end{array} \rrbracket} \forall E}{\frac{\forall Y G \llbracket B \rrbracket}{\forall Y G \llbracket B \rrbracket} \forall I} \forall E \\
\frac{}{G \llbracket B \rrbracket [F/Y]} \forall E
\end{array}
\equiv
\begin{array}{c}
\frac{\frac{\Gamma \llbracket A \rrbracket}{\mathcal{D}_1 \llbracket A \rrbracket} \quad \forall Y G \llbracket \begin{array}{c} A \\ \mathcal{D}' \\ B \end{array} \rrbracket}{G[F/Y] \llbracket B \rrbracket} \forall E
\end{array}
\stackrel{\text{i.h.}}{=}
\begin{array}{c}
\frac{\frac{\Gamma \llbracket \begin{array}{c} A \\ \mathcal{D}' \\ B \end{array} \rrbracket}{\mathcal{D}_1 \llbracket B \rrbracket} \quad \forall Y G \llbracket B \rrbracket}{G[F/Y] \llbracket B \rrbracket} \forall E
\end{array}
\stackrel{=^2_{\beta}}{=}$$

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To prove that \mathcal{D} is X -natural, we first reduce $\mathcal{D} \llbracket A \rrbracket$ and $\mathcal{D} \llbracket B \rrbracket$ to their β -normal forms $\mathcal{D}^\# \llbracket A \rrbracket$, $\mathcal{D}^\# \llbracket B \rrbracket$. Observe that $\mathcal{D}^\#$ is $\text{sp-}X$ and X -safe and therefore X -natural, and thus \mathcal{D} is X -natural too. \square

Remark 4.6. In [10] it is proved that derivations in \mathbf{NI} satisfy a more general property, called *dinaturality*. Contrary to X -naturality, which is tight to the notion of strictly positive occurrence of X in a formula, dinaturality takes into account also positive and negative occurrences of variables in formulas (in categorial terms, dinaturality accounts for both the covariant and the contravariant action of the functors associated to the formulas). This is the reason why, whereas only X -safe and $\text{sp-}X$ derivations can be shown to be natural, all X -safe derivations in \mathbf{NI} can be shown to be dinatural.

Our proof of theorem 4.8 is however significantly simpler than that in [10]. Indeed, the requirement of X occurring strictly positively has the consequence that formulas act as merely covariant functors and derivations correspond to natural (rather than dinatural) transformations, thus avoiding the well-known problem that dinatural transformations might not compose, see [3].

Observe that the X -safety requirement is essential to make the rule $\forall E$ (di-)natural (though, in a sense, X -safety makes $\forall E$ (di-)natural in a “trivial” way) and that neither theorem 4.8 nor its “dinatural” generalization can be extended to derivations that are not X -safe (a counter-example to dinaturality in \mathbf{NI}^2 can be found in [4]).

5 A natural extension of $\stackrel{2\triangleright}{\beta\eta}$

5.1 The (ε) -equation

The left-to-right orientation of the naturality condition discussed in the previous section has the flavour of a permutative conversion, in the sense that the derivation of B from A, Δ is permuted across the $\text{sp-}X$ and X -safe derivation of C from Γ . The connection between X -naturality and permutative conversions can be spelled out precisely by introducing the following class of formulas:

Definition 5.1 (quasi $\text{sp-}X$ formulas). Let $\forall \bar{Y}_i$ (for $1 \leq i \leq n$) denote finite (possibly empty) sequences of quantifiers whose variables are distinct from X . A formula of the form

$$\forall X \forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots))$$

is quasi $\text{sp-}X$ provided that F_1, \dots, F_n are $\text{sp-}X$ formulas.

Remark 5.1. The RP -translation of $A \vee B$ is of the form $\forall X (C \supset D \supset X)$, where C and D are $\text{sp-}X$. Thus it is a quasi $\text{sp-}X$ formula.

Remark 5.2. If $\forall XF$ is quasi $\text{sp-}X$ then $F \llbracket A \rrbracket \equiv \forall \bar{Y}_1 (F_1 \llbracket A \rrbracket \supset \forall \bar{Y}_2 (F_2 \llbracket A \rrbracket \supset \dots \supset \forall \bar{Y}_n (F_n \llbracket A \rrbracket \supset A) \dots))$.

We introduce now the following equation, where $\forall XF$ is quasi sp- X :

$$\begin{array}{c}
\begin{array}{c}
\mathcal{D}_1 \\
\hline
\forall XF \\
\hline
\forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)) \llbracket A \rrbracket \\
\hline
\forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)) \llbracket A \rrbracket \\
\hline
F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket A \rrbracket \quad F_1 \llbracket A \rrbracket \\
\hline
\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket A \rrbracket \quad \supset E
\end{array} \\
\vdots \\
\begin{array}{c}
\forall \bar{Y}_n (F_n \supset X) \llbracket A \rrbracket \\
\hline
F_n \supset X \llbracket A \rrbracket \quad \forall E \\
\hline
F_n \llbracket A \rrbracket \quad \supset E
\end{array} \\
\begin{array}{c}
A \\
\mathcal{D}_2 \\
B
\end{array}
\end{array}
=_{\varepsilon}
\begin{array}{c}
\begin{array}{c}
\mathcal{D}_1 \\
\hline
\forall XF \\
\hline
\forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)) \llbracket B \rrbracket \\
\hline
\forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)) \llbracket B \rrbracket \\
\hline
F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket B \rrbracket \quad F_1 \left[\begin{array}{c} A \\ \mathcal{D}_2 \\ B \end{array} \right] \\
\hline
\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket B \rrbracket \quad \supset E
\end{array} \\
=_{\varepsilon} \\
\vdots \\
\begin{array}{c}
\forall \bar{Y}_n (F_n \supset X) \llbracket B \rrbracket \\
\hline
F_n \supset X \llbracket B \rrbracket \quad \forall E \\
\hline
F_n \llbracket B \rrbracket \quad \supset E
\end{array} \\
B
\end{array}
\end{array}
\tag{\varepsilon}$$

We will call $=_{\varepsilon}$ the equivalence over NI^2 derivations generated by the closure under substitution, reflexivity and transitivity of the schema (ε) .

As an immediate consequence of theorem 4.8 we obtain a proposition analogous to 3.4 and 3.3 for the instances of (ε) in which the derivation \mathcal{D}_1 of $\forall XF$ is closed (we call these instances m-closed and we indicate with $=_{\varepsilon^{mc}}$ the equivalence relation induced by them):

Proposition 5.1. *If $\mathcal{D}', \mathcal{D}''$ are X -safe and $\mathcal{D}' =_{\varepsilon^{mc}} \mathcal{D}''$, then $\mathcal{D}' =_{\beta}^{2\supset} \mathcal{D}''$.*

Proof. Consider an instance of (ε) in which \mathcal{D}_1 is closed. Call \mathcal{D}' and \mathcal{D}'' the left-hand side and right-hand side of this instance of (ε) . We show $\mathcal{D}' =_{\beta}^{2\supset} \mathcal{D}''$.

Since \mathcal{D}_1 is closed, it β -reduces to a derivation $\mathcal{D}_1^\# \equiv \frac{\mathcal{D}_1^{\#'}}{F} \forall I$. Thus the left-hand side and right-hand side β -reduce to the following two derivations (by first reducing \mathcal{D}_1 to $\mathcal{D}_1^\#$ and then by getting rid of the resulting β -redex having $\forall XF$ as maximal formula):

$$\begin{array}{c}
\frac{\frac{\mathcal{D}_1^{\#'} \llbracket A \rrbracket}{F \llbracket A \rrbracket} \forall E}{F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket A \rrbracket} \supset E \\
\frac{}{\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket A \rrbracket} \supset E \\
\vdots \\
\frac{\frac{\forall \bar{Y}_n (F_n \supset X) \llbracket A \rrbracket}{F_n \supset X \llbracket A \rrbracket} \forall E}{\frac{A}{\mathcal{D}_2} \supset E} \supset E \\
\frac{}{B}
\end{array}
\quad
\begin{array}{c}
\frac{\frac{\mathcal{D}_1^{\#'} \llbracket B \rrbracket}{F \llbracket B \rrbracket} \forall E}{F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket B \rrbracket} \supset E \\
\frac{}{\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket B \rrbracket} \supset E \\
\vdots \\
\frac{\frac{\forall \bar{Y}_n (F_n \supset X) \llbracket B \rrbracket}{F_n \supset X \llbracket B \rrbracket} \forall E}{\frac{A}{\mathcal{D}_2} \supset E} \supset E \\
\frac{}{B}
\end{array}$$

Their $\beta^{2\supset}$ -equality follows from fact that, since \mathcal{D}_1 is closed and X -safe, so is $\mathcal{D}_1^\#$, and hence the derivation

$$\begin{array}{c}
\frac{\mathcal{D}_1^{\#'}}{F} \forall E \\
\frac{F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)}{\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots)} \supset E \\
\vdots \\
\frac{\frac{\forall \bar{Y}_n (F_n \supset X)}{F_n \supset X} \forall E}{X} \supset E
\end{array}$$

is $\text{sp-}X$, X -safe and thus X -natural by theorem 4.8. \square

Whereas m -closed instances of (ϵ) are included in $=_{\beta}^{2\supset}$, there are instances of (ϵ) which are not. This means that the equational theory on NI^2 -derivations induced by (ϵ) together with $(\beta\forall)$ and $(\beta\supset)$ (we indicate it with $=_{\beta\epsilon}^{2\supset}$) is a strict extension of the one induced by $(\beta\forall)$ and $(\beta\supset)$ alone. Actually, the same is true if one considers the extension $=_{\beta\eta\epsilon}^{2\supset}$ of $=_{\beta\eta}^{2\supset}$.

Proposition 5.2. *The equational theory $=_{\beta\eta\epsilon}^{2\supset}$ strictly extends $=_{\beta\eta}^{2\supset}$.*

Proof. For any derivation \mathcal{D}' of B from A we have that

$$\frac{\frac{\forall X(X \supset X)}{A \supset A} \forall E}{\frac{A}{\mathcal{D}'} \supset E} \supset E \equiv \frac{\frac{\forall X(X \supset X)}{X \supset X \llbracket A \rrbracket} \forall E}{\frac{A}{\mathcal{D}'} \supset E} \supset E =_{\epsilon} \frac{\frac{\forall X(X \supset X)}{X \supset X \llbracket B \rrbracket} \forall E}{\frac{A}{\mathcal{D}'} \supset E} \supset E \equiv \frac{\frac{\forall X(X \supset X)}{B \supset B} \forall E}{\frac{A}{\mathcal{D}'} \supset E} \supset E$$

However, whenever \mathcal{D}' is $\beta\eta$ -normal, the two member of the instance of (ε) just considered are distinct $\beta\eta$ -normal derivations and therefore (by the Church-Rosser property of the rewriting relation induced by β - and η -reductions) they are not $\beta\eta$ -equal. \square

Remark 5.3. It is worth stressing that this extension of $=_{\beta\eta}^{2\supset}$ is *consistent*: there exist derivations of the same conclusion from the same undischarged assumptions which are not identified by $=_{\beta\eta\epsilon}^{2\supset}$. This follows from the fact that the “dinatural” version of (ε) (see remark 4.6 above) is satisfied by most models of System F (see [3, 7, 10]).

Observe that via the Curry-Howard isomorphism, one can consider $=_{\beta\eta\epsilon}$ as an equivalence over λ -terms. Whereas (ε) consistently extends $\beta\eta$ -equality in System F, in the untyped case $=_{\beta\eta}$ is maximal (as a consequence of Böhm’s theorem) and thus $=_{\beta\eta\epsilon}$ is inconsistent over untyped lambda terms (i.e. $s =_{\beta\eta\epsilon} t$ for all terms s, t).

5.2 RP-translation and (ε) -equivalence

As observed in section 3, the RP-translation *does not* map $=_{\gamma_g}^{\vee}$ into either $=_{\beta}^{2\supset}$ or $=_{\beta\eta}^{2\supset}$, in the sense that $\mathcal{D}_1 =_{\gamma_g}^{\vee} \mathcal{D}_2$ implies neither $\mathcal{D}_1^* =_{\beta}^{2\supset} \mathcal{D}_2^*$ nor $\mathcal{D}_1^* =_{\beta\eta}^{2\supset} \mathcal{D}_2^*$. However, the RP-translation *does* map $=_{\gamma_g}^{\vee}$ into $=_{\beta\eta\epsilon}^{2\supset}$, in fact into $=_{\beta\epsilon}$ alone. More precisely,

Proposition 5.3 ($=_{\gamma_g}^{\vee} \mapsto =_{\beta\epsilon}$). *Let \mathcal{D}' and \mathcal{D}'' be, respectively, the left-hand side and right-hand side of $(\gamma_g\vee)$. One has $\mathcal{D}'^* =_{\beta\epsilon} \mathcal{D}''^*$.*

Proof. Since $(A \vee B)^* \equiv \forall X((A^* \supset X) \supset (B^* \supset X) \supset X)$, by the propositions 5.2 and 4.1 we have that

$$\begin{aligned}
\mathcal{D}'^* &\equiv \frac{\frac{\mathcal{D}^*}{(A \vee B)^*} \forall E \quad \frac{\frac{[A^*]^{n_1}}{\mathcal{D}_1^*} \supset I(n_1) \quad \frac{[B^*]^{n_2}}{\mathcal{D}_2^*} \supset I(n_2)}{A^* \supset C^*} \supset E}{\frac{(A^* \supset C^*) \supset (B^* \supset C^*) \supset C^*}{(B^* \supset C^*) \supset C^*} \supset E} \supset E \\
&\quad \frac{C^*}{\mathcal{D}_3^*} \supset E \\
&\quad D^* \\
&\equiv \frac{\frac{\mathcal{D}^*}{\forall X((A^* \supset X) \supset (B^* \supset X) \supset X)} \forall E \quad \frac{\frac{[A^*]^{n_1}}{\mathcal{D}_1^*} \supset I(n_1) \quad \frac{[B^*]^{n_2}}{\mathcal{D}_2^*} \supset I(n_2)}{A^* \supset X \llbracket C^* \rrbracket} \supset E}{\frac{A^* \supset X \llbracket C^* \rrbracket \supset B^* \supset X \llbracket C^* \rrbracket \supset C^*}{B^* \supset X \llbracket C^* \rrbracket \supset C^*} \supset E} \supset E \\
&\quad \frac{C^*}{\mathcal{D}_3^*} \supset E \\
&\quad D^*
\end{aligned}$$

[illegible]

Remark 5.4. By inspecting the proof of proposition 5.3 it is clear that for m-closed instances of $=_{\gamma_g}^{\vee}$, we have that $\mathcal{D}' =_{\gamma_g^{mc}}^{\vee} \mathcal{D}''$ implies $\mathcal{D}'^* =_{\varepsilon^{mc}\beta}^{\supseteq} \mathcal{D}''^*$. Thus, proposition 5.3 together with theorem 4.8 provides an alternative way to establish corollary 3.5.

Proposition 5.4 ($=_{\eta}^{\vee} \xrightarrow{*} =_{\beta\eta\epsilon}^{2\supset}$). *Let \mathcal{D}' and \mathcal{D}'' be, respectively, the left-hand side and right-hand side of $(\eta\vee)$. One has $\mathcal{D}'^* =_{\beta\eta\epsilon}^{2\supset} \mathcal{D}''^*$.*

[illegible]

Remark 5.5. By inspecting the proof of proposition 5.4 it is clear that for m-closed instances of $=_{\check{\gamma}}$, we have that $\mathcal{D}' =_{\check{\gamma}_{mc}}^{\check{\gamma}} \mathcal{D}''$ implies $\mathcal{D}'^* =_{\beta\eta^{mc}mc}^{2\supset} \mathcal{D}''^*$. Thus, proposition 5.4 together with theorem 4.8 (and with remark 3.1) provides an alternative way to establish corollary 3.6.

The results above can be given for the intuitionistic connectives \wedge and \perp as well, by using their RP-translations $\forall X((A \supset B \supset X) \supset X)$ and $\forall XX$. For instance, the generalized permutation for \perp discussed in section 2 ($\gamma_g \perp$) is mapped by the RP-translation onto the following instance of (ϵ):

Actually, these results can be generalized to the much wider class of connectives introduced by Schroeder-Heister in the context of his natural extension of natural deduction with rules of arbitrary level [17].

According to [15], given r introduction rules for the connective \dagger , which have the following general form (for $1 \leq h \leq r$):

$$\frac{[A_{11}^h] \dots [A_{k_1 1}^h] \quad \dots \quad [A_{1n_h}^h] \dots [A_{k_{n_h} n_h}^h]}{\dagger(C_1 \dots C_m) \quad \dagger I_h} \quad \begin{array}{c} B_1^h \quad \dots \quad B_{n_h}^h \end{array}$$

where all the A_{ij}^h and B_j^h are identical with one of the C_i ($1 \leq l \leq k_j$, $1 \leq j \leq n_h$ and $1 \leq i \leq m$). Applications of the rule can discharge the assumptions of the form A_{lj}^h ($1 \leq l \leq k_j$) in the derivation of the premise B_j^h ($1 < j < n_h$).

Given r introduction rules $\dagger I_1, \dots, \dagger I_r$ for the connective \dagger of the form above, a unique elimination rule construed after the model of disjunction which “inverts” (in the sense of Lorenzen’s inversion principle) this collection of introduction rules is the following:

$$\frac{\dagger(C_1 \dots C_m) \quad \begin{array}{c} [R_1^1] \dots [R_{n_1}^1] \\ X \end{array} \quad \dots \quad \begin{array}{c} [R_1^r] \dots [R_{n_r}^r] \\ X \end{array}}{X} \quad \dagger E \quad 5$$

where X is fresh and each R_j^h discharged by the $\dagger E$ rule corresponds to the j -th premise of the \dagger -intro $_h$ rule. I.e., $R_j^h = A_{1j}^h \supset \dots \supset A_{k_j j}^h \supset B_j^h$ if $k > 0$; $R_j^h = B_j^h$ otherwise.

The RP-translation of $\dagger(C_1 \dots C_m)$ is given by the formula

$$\dagger(C_1 \dots C_m)^* := \forall X((R_1^1 \supset \dots \supset R_{n_1}^1 \supset X) \supset \dots \supset (R_1^r \supset \dots \supset R_{n_r}^r \supset X) \supset X)$$

As all formulas of the form $R_1^h \supset \dots \supset R_{n_h}^h \supset X$ are sp- X , the formula $\dagger(C_1 \dots C_m)^*$ is quasi sp- X , and thus results analogous to proposition 5.3 and 5.4 can be established for all such connectives as well.

In [17], the left-iterated implications in the elimination rules are eliminated by enriching the structural means of expression of natural deduction by allowing not only formulas but also (applications of) rules to be assumed in the course of a derivation and by allowing rule to discharge not only formulas but also previously assumed rules. Once the structural device of rule-discharge is introduced, nothing prohibits its use in introduction rules, thereby yielding a yet richer class of connectives definable by means of introduction and elimination rules. In [18], the structural means of expression have been further enriched by admitting a form of structural quantification, in terms of which, for instance, the introduction rule for negation can be formulated as:

$$\frac{\left(\begin{array}{c} [A] \\ X \end{array} \right)_X}{\neg A}$$

where the notation $()_X$ indicates that X plays the same role of the *eigenvariable* X in the second order $\forall I$. By iterating structural quantification and higher-level discharge it is

easy to construct for any quasi sp- X NI^2 -formula F a collection of introduction rules for a connective \dagger such that $\dagger(C_1 \dots C_m)^* \equiv F$. Consequently, the results presented in this section can be extended to arbitrary connectives of the calculus of higher-level rules with propositional quantification.

6 Atomic polymorphism

In recent work [5, 6] Ferreira and Ferreira investigated a variant of the RP-translation which maps NI^\vee -derivations into derivations of a subsystem NI_{at}^2 of NI^2 where the rule $\forall E$ is restricted to *atomic* instantiation⁴. Ferreira and Ferreira refer to the system as \mathbf{F}_{at} of “atomic polymorphism”. Remark that the system NI_{at}^2 enjoys the sub-formula property.

The main ingredient of their translation is the result, that they call *instantiation overflow*, showing that the unrestricted version of the rule $\forall E$ is derivable in NI_{at}^2 whenever the premise of $\forall E$ is $(A \vee B)^*$ (or $\forall X(A \supset B \supset X)$ or $\forall XX$).

Slightly reformulating Ferreira and Ferreira’s insight, we can define the following mapping of NI^2 -derivations obtained by RP-translating an NI^\vee -derivations into NI_{at}^2 -derivations.

Definition 6.1 (Instantiation overflow). *If \mathcal{D} is the Russell-Prawitz translation of some NI^\vee -derivation, then \mathcal{D}^\downarrow is defined by induction on \mathcal{D} . We only consider the case in which the last rule \mathcal{D} is $\forall E$ since all other rules are translated in a trivial way. In this case observe that*

$$\mathcal{D} \equiv \frac{\mathcal{D}' \quad \frac{\forall X(A \vee B)}{(A \vee B)[F/X]} \forall E}{\text{where, as before, } A \vee B \text{ abbreviates } (A \supset X) \supset (B \supset X) \supset X}.$$

We define \mathcal{D}^\downarrow by a sub-induction on F .

- If $F \equiv Y$ then $\mathcal{D}^\downarrow = \frac{\mathcal{D}'^\downarrow \quad \frac{\forall X(A \vee B)}{(A \vee B)[Y/X]} \forall E}{\text{where, as before, } A \vee B \text{ abbreviates } (A \supset X) \supset (B \supset X) \supset X}.$
- If $F \equiv \forall YD$ then

$$\mathcal{D}^\downarrow \equiv \frac{\left(\frac{\mathcal{D}'^\downarrow \quad \frac{\forall X(A \vee B)}{(A \vee B)[D/X]} \forall E}{(B \supset D) \supset D} \right)^\downarrow \quad \frac{\frac{A \supset \overset{k_1}{\forall YD} \quad \overset{o}{A}}{\frac{\forall YD}{D} \forall E} \supset E \quad \frac{\frac{B \supset \overset{k_2}{\forall YD} \quad \overset{o}{B}}{\frac{\forall YD}{D} \forall E} \supset E}{\frac{\frac{D}{\forall YD} \forall I}{(B \supset (\forall YD)) \supset (\forall YD)} \supset I (o) \quad \frac{D}{\forall YD} \forall I}{\frac{(B \supset (\forall YD)) \supset (\forall YD)}{(A \vee B)[\forall YD/X]} \supset I (k_2) \quad \frac{D}{\forall YD} \forall I} \supset E$$

If $F \equiv C \supset D$ then

⁴Similar restrictions have been investigated in [11].

$$\begin{array}{c}
\mathcal{D}^\downarrow \equiv \frac{\left(\frac{\mathcal{D}'}{\forall X(A \vee B)} \vee E \right)^\downarrow}{(B \supset D) \supset D} \quad \frac{\frac{A \supset (\overset{k_1}{C} \supset D) \quad \overset{o}{A}}{C \supset D} \supset E \quad \overset{m}{C}}{\frac{D}{A \supset D} \supset I(o)} \supset E \quad \frac{B \supset (\overset{k_2}{C} \supset D) \quad \overset{o}{B}}{C \supset D} \supset E \quad \overset{m}{C}}{\frac{D}{B \supset D} \supset I(o)} \supset E \\
\frac{\frac{D}{C \supset D} \supset I(m)}{(B \supset (C \supset D)) \supset (C \supset D)} \supset I(k_2) \\
\frac{(B \supset (C \supset D)) \supset (C \supset D)}{(A \vee B) \llbracket C \supset D/X \rrbracket} \supset I(k_1)
\end{array}$$

If \mathcal{D} is an NI^\vee -derivation, we call the NI_{at}^2 -derivation $\mathcal{D}^{*\downarrow}$ the Ferreira-Ferreira translation of \mathcal{D} (which we abbreviate to FF-translation).

Ferreira and Ferreira showed that if two NI^\vee -derivations are γ -equivalent their FF-translations are $\beta\eta$ -equivalent NI_{at}^2 -derivations.

In fact, an analogous result can be shown for $(\eta\vee)$ as well. That is, if two NI^\vee -derivations are η -equivalent, then their FF-translations are $\beta\eta$ -equivalent NI_{at}^2 -derivations. More precisely,

Proposition 6.1 ($=_{\eta}^\vee \xrightarrow{*} =_{\beta\eta}^{2\supset}$). Let \mathcal{D}' and \mathcal{D}'' be, respectively, the left-hand side and right-hand side of $(\eta\vee)$. One has $\mathcal{D}'^{*\downarrow} =_{\beta\eta}^{2\supset} \mathcal{D}''^{*\downarrow}$.

Proof. The FF-translation of \mathcal{D}' (depicted in figure 1) is not in β -normal form; in particular, it $\beta\eta^{2\supset}$ -reduces to $\mathcal{D}^{*\downarrow} \equiv \mathcal{D}''^{*\downarrow}$. □

These results suggest the existence of a tight connection between the FF-translation and (ε) . To make it explicit, we first reformulate the definition of \downarrow using the notion of C-expansion as follows:

Definition 6.2 (Instantiation overflow, alternative definition). If \mathcal{D} is the Russell-Prawitz translation of some NI^\vee -derivation, then \mathcal{D}^\downarrow is defined by induction on \mathcal{D} . We only consider the case in which the last rule \mathcal{D} is $\forall E$ since all other rules are translated in a trivial way. In this

case observe that $\mathcal{D} \equiv \frac{\mathcal{D}'}{\frac{\forall X(A \vee B)}{(A \vee B) \llbracket F/X \rrbracket} \forall E}$ for some NI^2 -formula $F \equiv \forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset Z) \dots))$. We assume by proposition 4.1 that \mathcal{D} is X-safe, i.e. that X does not occur in F. Using the notation introduced in remark 4.5 we define:

Table 1: The FF-translation of the left-hand member of $(\eta \vee)$

$$\begin{array}{c}
 \frac{\mathcal{G}^{!*}}{\frac{(A \vee B)^*}{A^* \vee B^*} \vee E} \quad \frac{\frac{A^* \supset (A \vee B)^* \quad A^* \supset X}{(A \vee B)^* \supset E} \quad \frac{A^* \supset X}{(B^* \supset X) \supset X} \supset E \quad \frac{B^* \supset X}{B^* \supset X} \supset E}{\frac{(B^* \supset X) \supset X}{(B^* \supset X) \supset X} \supset E} \quad \frac{\frac{B^* \supset (A \vee B)^* \quad B^* \supset X}{(A \vee B)^* \supset E} \quad \frac{A^* \supset X}{(B^* \supset X) \supset X} \supset E \quad \frac{B^* \supset X}{B^* \supset X} \supset E}{\frac{(B^* \supset X) \supset X}{(B^* \supset X) \supset X} \supset E} \quad \frac{\frac{X}{A^* \supset X} \supset I (2)}{\frac{(B^* \supset X) \supset X}{(B^* \supset X) \supset X} \supset E} \quad \frac{\frac{X}{B^* \supset X} \supset I (6)}{\frac{(B^* \supset X) \supset X}{(B^* \supset X) \supset X} \supset E} \\
 \frac{\frac{X}{(B^* \supset X) \supset X} \supset I (4)}{\frac{A^* \vee B^*}{(A \vee B)^*} \vee I} \supset I (3) \quad \frac{\frac{A^* \vee B^*}{(A \vee B)^*} \vee I}{(B^* \supset (A \vee B)^*) \supset (A \vee B)^*} \supset I (5) \quad \frac{(B^* \supset (A \vee B)^*) \supset (A \vee B)^*}{A^* \vee B^*[(A \vee B)^*/X]} \supset I (1) \\
 \frac{A^* \vee B^*[(A \vee B)^*/X]}{(B^* \supset (A \vee B)^*) \supset (A \vee B)^*} \supset E \quad \frac{(B^* \supset (A \vee B)^*) \supset (A \vee B)^*}{(A \vee B)^*} \supset E \\
 \frac{\frac{A^* \supset X \quad A^* \supset X}{X} \supset E \quad \frac{X}{(B^* \supset X) \supset X} \supset I}{\frac{A^* \vee B^*}{(A \vee B)^*} \vee I} \supset I (7) \quad \frac{\frac{B^* \supset X \quad B^* \supset X}{X} \supset E \quad \frac{X}{(B^* \supset X) \supset X} \supset I}{\frac{A^* \vee B^*}{(A \vee B)^*} \vee I} \supset I (9) \\
 \frac{\frac{A^* \supset (A \vee B)^*}{A^* \supset (A \vee B)^*} \supset I (8) \quad \frac{A^* \supset (A \vee B)^*}{B^* \supset (A \vee B)^*} \supset I (10)}{\frac{(A \vee B)^*}{(A \vee B)^*} \supset E} \supset E
 \end{array}$$

$$\begin{array}{c}
\mathcal{D}' \downarrow \\
\frac{\forall X(A \vee B)}{(A \vee B)[[Z/X]]} \forall E \quad \frac{A \supset X \quad \begin{array}{|c|} \hline F \\ \mathcal{E}_F^e \\ Z \\ \hline \end{array}^{k_1}}{(B \supset Z) \supset Z} \supset E \quad \frac{B \supset Z \quad \begin{array}{|c|} \hline F \\ \mathcal{E}_F^e \\ Z \\ \hline \end{array}^{k_2}}{Z} \supset E \\
\mathcal{D}' \downarrow \equiv \frac{\frac{\frac{F}{(B \supset F) \supset F} \supset I(k_2)}{(A \vee B)[[F/X]]} \supset I(k_1)}{\mathcal{E}_F^i(m_1, \dots, m_n)}
\end{array}$$

The reformulation of the embedding \downarrow can be easily generalized to derivations in which the premises of all applications of $\forall E$ are quasi sp-X. Thus, for each such derivation we can find an NI_{at}^2 -derivation of the same conclusion from the same undischarged assumptions:

Definition 6.3 (generalized instantiation overflow). *Let \mathcal{D} be an sp-X NI_{at}^2 -derivation in which all premises of applications of $\forall E$ are quasi sp-X. The derivation \mathcal{D}' in NI_{at}^2 is defined by induction on \mathcal{D} . As before, we only consider the case in which the last rule \mathcal{D} is $\forall E$ since all other rules are translated in a trivial way. In this case, we can assume that $\mathcal{D} \equiv \frac{\mathcal{D}'}{\frac{\forall XF}{F[G/Y]} \forall E}$ where $F = \forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots))$ is a quasi sp-X formula, and X does not occur in $G = \forall \bar{Z}_1 (G_1 \supset \forall \bar{Z}_2 (G_2 \supset \dots \supset \forall \bar{Z}_m (G_m \supset Z) \dots))$. Using the notation introduced in remark 4.5 we thus have:*

$$\begin{array}{c}
\mathcal{D}' \downarrow_s \\
\frac{\forall XF}{F[G/X]} \forall E \\
\frac{F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) [Z]}{\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) [Z]} \supset E \\
\vdots \\
\frac{\forall \bar{Y}_n (F_n \supset X) [Z]}{F_n \supset X [Z]} \forall E \\
\frac{Z}{\mathcal{E}_G^i} \\
\frac{G}{F_n \supset G} \supset I(k_n) \\
\frac{F_n \supset G}{\forall \bar{Y}_n (F_n \supset G)} \forall I \\
\vdots \\
\frac{(F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) [G/X]}{F[G/X]} \supset I(k_1) \\
\forall I
\end{array}$$

The following holds:

Theorem 6.2. Let \mathcal{D} be an X -safe, sp - X derivation in which the premises $\forall XF$ of all applications of $\forall E$ are quasi sp - X . $\mathcal{D} \stackrel{2}{\Rightarrow}_{\eta\epsilon} \mathcal{D}' \downarrow_s$

Proof. The proof is by induction on \mathcal{D} . If \mathcal{D} ends with an application of either $\supset I$, $\supset E$, or $\forall I$, then it is enough to apply the induction hypothesis to the the immediate sub-derivations of \mathcal{D} .

If \mathcal{D} ends with an application of $\forall E$, i.e. $\mathcal{D} \equiv \frac{\mathcal{D}'}{\frac{\forall XF}{F[G/X]} \forall E'}$, we have that $F \equiv \forall \bar{Y}_1 (F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots))$ is a quasi sp - X formula, and X does not occur in $G = \forall \bar{Z}_1 (G_1 \supset \forall \bar{Z}_2 (G_2 \supset \dots \supset \forall \bar{Z}_m (G_m \supset Z) \dots))$. We thus have:

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$$\begin{array}{c}
\frac{\mathcal{D}' \downarrow_g}{\frac{\forall XF}{F[Z/X]} \forall E} \quad \frac{F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket Z \rrbracket}{\forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket Z \rrbracket} \forall E \quad \frac{F_1 \supset \left[\begin{array}{c} G \\ \mathcal{E}_G^e \\ Z \end{array} \right]^{k_1}}{\vdots} \supset E \\
\vdots \\
\frac{\frac{\text{i.h.}}{=_{\eta\epsilon}^2} \quad \frac{\forall \bar{Y}_n (F_n \supset X) \llbracket Z \rrbracket}{F_n \supset X \llbracket Z \rrbracket} \forall E \quad \frac{F_n \supset \left[\begin{array}{c} G \\ \mathcal{E}_G^e \\ Z \end{array} \right]^{k_n}}{\vdots} \supset E \equiv \mathcal{D} \downarrow_g}{\frac{Z}{\mathcal{E}_G^i} \quad \frac{G}{F_n \supset G} \supset I (k_n) \quad \frac{F_n \supset G}{\forall \bar{Y}_n (F_n \supset G)} \forall I \quad \vdots \quad \frac{(F_1 \supset \forall \bar{Y}_2 (F_2 \supset \dots \supset \forall \bar{Y}_n (F_n \supset X) \dots) \llbracket G/X \rrbracket}{F \llbracket G/X \rrbracket} \supset I (k_1) \quad \forall I} \supset E \equiv \mathcal{D} \downarrow_g
\end{array}$$

□

Remark 6.1. What definition 6.3 and theorem 6.2 establish is that for the fragment of NI^2 consisting of X -safe derivation in which the rule $\forall E$ is restricted to quasi sp- X premises the sub-formula holds, in the sense that for any derivation in this fragment there is one in NI_{at}^2 of the same conclusion from the same undischarged assumptions. By normalization there is also a β -normal one, and normal derivations in NI_{at}^2 enjoy the sub-formula property.

As the mapping \downarrow of Ferreira and Ferreira is just the instance of \downarrow_g obtained by taking F to be $(A \supset X) \supset (B \supset X) \supset X$ (or $(A \supset B \supset X$ and X in the case of \vee and \perp), we therefore have the following:

Corollary 6.3. $\mathcal{D} =_{\eta\epsilon}^2 \mathcal{D} \downarrow$.

Corollary 6.4. If $\mathcal{D}_1^* \downarrow =_{\beta\eta}^2 \mathcal{D}_2^* \downarrow$ then $\mathcal{D}_1^* =_{\beta\eta\epsilon} \mathcal{D}_2^*$.

Remark 6.2. The left-to right orientation of (ϵ) can thus be viewed as a form of permutation that together with η -expansion allows to atomize the application of $\forall E$, provided the premise of the rule application is quasi sp- X . Conversely, the FF-translation \mathcal{D} can be viewed as consisting of a (huge) series of η -expansions followed by a ϵ -permutation applied to the RP-translation of \mathcal{D} .

One may therefore be willing to argue that there is a trade-off between the RP-translation of NI^\vee into full NI^2 and the FF-translation into NI_{at}^2 .

In the case of the FF-translation, one must adopt a less natural translation (consisting in the combination of the RP-translation and of the embedding into NI_{at}^2) which does not directly preserve β -reduction (as not only β^\vee - but also β^\supset -equivalent NI^2 -derivations are mapped onto merely $\beta\eta$ -equivalent NI_{at}^2 -derivations). This is the price for getting permutative conversions and η -reduction, in a sense, for free.

In the case of the original RP-translation, on the other hand, the translation of derivations is more straightforward and one has a direct preservation of β -reduction. However, η and γ equivalences are retrieved only at the price of extending the equational system of NI^2 with (ϵ) .

In spite of this, the two approaches are not equivalent. In fact, it is not the case that, if $\mathcal{D}_1^* =_{\beta\eta\epsilon} \mathcal{D}_2^*$, then $\mathcal{D}_1^{*\downarrow} =_{\beta\eta}^2 \mathcal{D}_2^{*\downarrow}$. Using ϵ , one can RP-translate both γ and its generalization γ_g . On the other hand, the FF-translations of γ_g equivalent derivations are, in general, not $\beta\eta$ -equivalent in NI_{at}^2 . This fact is a consequence of the strong normalization of NI^2 (which clearly implies that for NI_{at}^2) and of the fact that the permutation induced by (γ_g^\vee) is non-strongly normalizing.

Thus, we would tend to disagree with Ferreira and Ferreira “proposal: embed the intuitionistic predicate calculus into $[\text{NI}_{at}^2]$, where there are no bad rules. We tentatively suggest that this is the right way to see the connectives $\perp, \vee [\dots]$ in Structural Proof Theory: through the lens of the above embedding.” ([5], p. 68). The reason for disagreement is that Structural Proof Theory should be concerned with identity of proofs, and the notion of identity of proof for NI^\vee cannot be fully rendered using the FF-translation of NI^\vee into NI_{at}^2 .

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